Late states of incompressible 2D decaying vorticity fields

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Abstract

Two-dimensional decaying turbulent flow is known to approach apparently stable states after a long time evolution. A few theories and models have been so far proposed to account for this relaxation. In this paper, we compare results of numerical experiments with the predictions of these theories to assess their applicability. We study the long time decay of initially multilevel vorticity fields on the periodic box, and characterize the outcoming final states. Our final states do not match the predictions of the theories; a broader variety of dipole profiles, as well as nonstationary final states are found. The problem of the robustness of the relaxational state with respect to variations of the Reynolds number and different numerical resolution is addressed. The observed configurations also do not necessarily possess the maximal energy, in contrast to what is anticipated by some of the theories. We are led to conclude that the mixing of the vorticity is generally not ergodic, and that some metastable configurations may inhibit the attainment of an equilibrium state.

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1 Introduction and motivations

The decay of the vorticity field of a two-dimensional incompressible fluid, which obeys to the unforced Navier-Stokes equation, shows several interesting features. Stable large scale structures may form, organize themselves on the scale of the accessible domain, and survive for a long time before being ultimately damped by the dissipation. Occurrences of such structures are observed in numerical and laboratory experiments, as well as in planetary scale flows. We will call from here on such configurations "final"; by that we mean subjected only to viscous decay and no more to filamentation and mixing. Provided that the Reynolds number of the flow is large enough, the timescales for these two processes can be very different. We are concerned with the appearance of final states emerging from arbitrary initial conditions, with their robustness and their predictability. Most of the numerical simulations presented in the literature start from random initial conditions, whose Fourier spectra decay variously in k, with uncorrelated phases. In such cases, intermediate states with few isolated vortex cores are observed. In the traditional scenario (Benzi et al. 1988), the progressive merging of likely signed vortex cores is then observed. Several papers in the literature deal with the merging process, either analyzing the evolution in time of the vortex census, or modelling the process with simplified dynamics. Less attention has been so far devoted to the actual characteristics of the latest state of the field.

A few explanations for the occurrence of final states have been proposed so far. We will refer specifically to those theories, which are formulated in the physical space rather than in the Fourier one: the minimum enstrophy assumption and the maximal entropy state statistical theory (Miller et al. 1992,Robert and Sommeria 1991). Some numerical evidence has also been invoked to reconnect computed results with the Joyce-Montgomery equation, which is originally derived for the mean field limit of a system of point vortices (Montgomery et al. 1993).

We consider several cases of decay of vorticity fields. All cases start from initial conditions which should be generic and appropriate for the application of the final state theories. Our results contrast with the conclusion that an universal state emerges out of the relaxation, without memory of the dynamical path which led to it. In fact, we even see that the final 'state' may be unsteady: some fields attain a late time configuration which is not stationary. Such late fields may move quasiperiodically and steadily, being just slowly damped by the (arbitrary small) dissipation.

This paper is organized as follows: in section 2 we briefly review the treatment of the problem of two-dimensional turbulent decay, highlighting the different variational problems which are solved to find the final state, and the functional relations $\omega(\psi)$ derived. In section 3 we describe the numerical experiments per-

formed. Finally, in section 4 we comment on the open aspects which require further investigation. A number of remarks which are related to the maximization of the energy and its relevance for the final state are left for the Appendix.

2 Review of the available theories

Let us first recall the notation. As is well known, the equation of motion for the vorticity $\omega(\mathbf{x})$ is written as

$$\omega_t(\mathbf{x}) = -\mathbf{v}(\mathbf{x}) \cdot \nabla\omega(\mathbf{x}) + \nu\nabla^2\omega(\mathbf{x}) = J(\omega(\mathbf{x}), \psi(\mathbf{x})) + \nu\nabla^2\omega(\mathbf{x})\,. \quad (1)$$

In (1), ψ denotes the streamfunction, obtained from $\omega(\mathbf{x})$ by means of the Green function G of the Laplacian operator

$$\psi(\mathbf{x}) = \int G(\mathbf{x}', \mathbf{x})\omega(\mathbf{x}')d^2\mathbf{x}\,. \quad (2)$$

The integral is carried on the fluid domain with the proper boundary conditions, so that $\nabla^2\psi = -\omega$. Introducing the notation $\nabla^\perp = (\partial_y, -\partial_x)$, we may write $\nabla^\perp\psi = \mathbf{v}$ and $J(\omega, \psi) = \nabla\omega \cdot \nabla^\perp\psi$. The energy of the field is defined as

$$E = \frac{1}{2}\int \psi(\mathbf{x})\omega(\mathbf{x})d^2\mathbf{x}\,, \quad (3)$$

and the moments of the vorticity as

$$Q_l = \frac{1}{2}\int \omega^l(\mathbf{x})d^2\mathbf{x}\,, \quad (4)$$

with l a positive integer. The quantity $Q_2(\equiv Q)$ is traditionally called the enstrophy. Another global quantity which is defined is the palinstrophy

$$P = \frac{1}{2}\int \left[\nabla\omega(\mathbf{x})\right]^2 d^2\mathbf{x}\,. \quad (5)$$

These quantities evolve in time according to:

$$\begin{aligned} E_t = &\quad -2\nu Q\,, \qquad Q_t = -2\nu P\,, \\ Q_{l,t} = &-l(l-1)\nu\int \omega(\mathbf{x})^{l-2}\left[\nabla\omega(\mathbf{x})\right]^2 d^2\mathbf{x}\,. \end{aligned} \quad (6)$$

E and all Q_l are constants of motion if $\nu = 0$. In the limit of vanishing viscosity, E is a constant of motion, but Q may not be, because P can get larger inversely proportional to ν.

We refer to theories which predict the final state as the most probable outcome of the decay. In a way or the other, all these models assume a distinction between the fully detailed dynamics expressed by equation (1), and that of a reduced set of macroscopically observable quantities. The evolution of the field is seen as a process in which the initial information is lost in some way. As a deliberate simplification, the final state is sought as the one which is fully described only by a few macroscopical constraints, which are often called "rugged invariants". Such theories treat the viscous, finite resolution problem, as one in which some quantities are "better" conserved than the rest, in place of the infinite set of the inviscid case. There is indeed some ambiguity, and properly speaking the case $\nu = 0$ is different from the limit $\nu \to 0$, since in the former infinitely steep gradients might form. Where possible, methods of statistical mechanics are applied, and some extremum principle is invoked. A comprehensive review of the various positions can be found elsewhere (e.g. Miller et al. 1992); here we recall them briefly, and discuss their conclusions.

2.1 Equilibrium Fourier spectra

A first group of theories is formulated in the Fourier space. The older Kraichnan–Batchelor–Leith statistical theory (Kraichnan 1967) predicts a Fourier spectrum $E(k) \sim k^{-3}$, relying on the assumption of the locality of the interactions among the components. An improvement by Kraichnan (1971), based on the test-field-model closure approximation, corrected this spectrum to $E(k) \sim k^{-3} (\ln k/k_0)^{\frac{1}{3}}$. While some numerical simulations (Kida et al. 1988,Brachet et al. 1988) support these spectra, quite different spectra have been observed by others (see for example McWilliams 1984) The reason of the discrepancy is not clear, though the formation of stable structures may take a key role.

A later theory due to Kraichnan (Kraichnan and Montgomery 1980,Carnevale 1982) proposes a statistical mechanics for the energies of the Fourier components. An ultaviolet cutoff in k has to be enforced. Only E and Q are assumed to be constants of motion, and are fixed as constraints. No correlation is assumed between the phases of $\omega(k)$, and no other moment of the vorticity is conserved. This theory predicts a statistical equilibrium spectrum

$$E(k) = \frac{1}{\beta k^2 + \alpha}, \qquad Q(k) = \frac{k^2}{\beta k^2 + \alpha}, \tag{7}$$

with arbitrary constants α and β. The agreement of these spectra with those

coming from numerical simulations, and especially with ours, is controversial.

2.2 Point vortex systems

A second line of reasoning considers the statistical properties of an ensemble of point vortices. The rationale for connecting vorticity fields with such systems is that a system of point vortices approximates weakly, in the continuum limit, the Euler equation (Caglioti et al. 1992,Eyink and Spohn 1993,Campbell and O'Neil 1991); the full Navier-Stokes equation can be emulated by point vortices which diffuse with an additional Brownian motion (Chorin 1994). An entropy of the system is introduced and maximized. In the mean field limit, a differential equation is derived for the equilibrium configuration (Montgomery and Joyce 1974,Kida 1975,Pointin and Lundgre

$$\omega_0(\mathbf{x}) = -\nabla^2\psi_0(\mathbf{x}) = c_1\,\mathrm{e}^{-\beta\psi_0(\mathbf{x})} - c_2\,\mathrm{e}^{\beta\psi_0(\mathbf{x})}\ . \tag{8}$$

This provides us with a first example of an equation which relates functionally ω_0 and ψ_0. As is known, the functional dependence implies the stationarity of the motion, in the case of null dissipation. In the special case of an equal number of opposite charged positive and negative vortices, the Joyce-Montgomery equation reduces to the sinh-Poisson equation

$$\omega_0(\mathbf{x}) = \lambda^2 \sinh(|\beta|\psi_0(\mathbf{x}))\,. \tag{9}$$

This equation has been furthermore studied referring to the inverse scattering theory for the sin-Gordon equation (Ting et al. 1987). Solutions possessing simple scattering spectra can be constructed, but no dynamical analysis, beyond a comparison of shapes, was done.

Montgomery et al. (1993) give an interpretation of the problem which is somehow related. They propose a decomposition of the vorticity field in four non-physical positive subfields. An entropy of the form

$$S = \int \omega_i \ln \omega_i d^2\mathbf{x} \tag{10}$$

is then maximized individually for each subfield, with the proper constraints on the total energy and vorticity. The remaining arbitrary constants are fitted to the results of a single high resolution, high Reynolds number Navier–Stokes numerical simulation. For that case, they achieve a good fit of the $\psi(\omega)$ scatter-plot at late times. For our purposes, it suffices to note that their final relation implies

$$\omega_0(\mathbf{x}) = c_1\,\mathrm{e}^{-\beta\psi_0(\mathbf{x})} - c_2\,\mathrm{e}^{\beta\psi_0(\mathbf{x})} + c_3\,, \tag{11}$$

which is an elaboration of (8), and can be assimilated to equation (21) in the case of 3 levels.

2.3 Minimum enstrophy principle

The identification of the final state as the one with lowest enstrophy dates back to Bretherton and Haidvogel (1976). They argued that while the energy can almost be conserved by a good numerical scheme, the vorticity filamentates progressively and smoothes out. Arguments related to the universality of the energy-enstrophy cascades, as in the Kraichnan–Batchelor–Leith theory, would predict for instance a behavior of $Q(t) \sim t^{-2}$ (Carnevale et al. 1992,Bartello and Warn 1996). The idea of a faster decay of the enstrophy with respect to the energy is often referred to as the "selective decay hypothesis". The final state is consequently found variationally, by minimizing Q with constrained E. According to this hypothesis, axisymmetric vortex shapes on the infinite plane can be calculated (Leith 1984). Time asymptotic estimates for closed square box solutions are discussed by van Groesen (1988).

In the case of doubly periodical boundary conditions, it is straightforward to find a solution. Imposing

$$\frac{\delta}{\delta\omega}\frac{1}{2}\int(\omega^2 - \lambda\psi\omega)d^2\mathbf{x} = 0\,, \tag{12}$$

we obtain

$$\omega_0(\psi_0) = \lambda\psi_0\,, \tag{13}$$

which on the periodic square admits solutions of the form $\omega_0(\mathbf{x}) = \sum \omega_\mathbf{k}\, \mathrm{e}^{\imath\mathbf{k}\cdot\mathbf{x}}$, with $|\mathbf{k}|^2 = \lambda$ equal to a squared integer. This implies that $Q = \lambda E$. For a given energy the minimal enstrophy is then achieved for $\lambda = 1$, and the general solution becomes $\omega_0(\mathbf{x}) = \omega_1 \cos(x + a) + \omega_2 \cos(y + b)$, with ω_1, ω_2, a and b being arbitrary constants. Linear combinations of such sinusoidal solutions with different k (i.e. complete Fourier series) do *not* satisfy the requirement. In the numerical experiments we find final states of completely different forms. We remark however that this principle was introduced for flows with additional "topographic" terms in (1), and we do not exclude that it may provide realistic results in cases where these are dominant.

2.4 Vortex censuses and punctuated dynamics

In a number of papers, appeared around 1990, the late lowering of the enstrophy is solely explained as a result of progressive vortex mergings (McWilliams 1990,Matthaeus et al. 199
These papers consider situations with an intermediate time dynamics dominated by many well separated vortex cores, which behave approximately like point vortices. The subsequent evolution is schematized by a progressive collapse and merging of these objects. Statistics of the number N of cores in time, models for the probability of merging are examined, and eventually lead to different scalings of $Q(t)$. The final state is assumed to be a dipole, and its properties are sought by scaling the relevant quantities down to $N = 2$. A popular model is the punctuated Hamiltonian system (Carnevale et al. 1991), which is the traditional point vortex model fitted with a nonconservative merging as vortices get close enough. Such models can be further elaborated accounting for extended cores (Riccardi et al. 1995).

2.5 Maximum entropy theory

The reasoning is based on the combinatorics of infinitesimal vorticity patches, at a scale smaller than that which determines the energy of the configuration. We follow the notation of Miller et al. (1992), rather than the equivalent one of Robert and Sommeria (1991). The theory is indeed intended only for the Euler equation; Weichman (1993) proposes an additional argument in order to include the viscosity in connection with the underresolution, which seems incorrect. The theory mimics the statistical mechanics of a many particle system. A given distribution of (infinitesimally grained) vorticity is assimilated to a microstate; at the macroscopical level, only a coarse averaged vorticity $\overline{\omega}(\mathbf{x})$ can be observed. The fine scale structure is remembered by introducing a local probability distribution $n(\mathbf{x}, \sigma)d\sigma$ of vorticity, which says how large is the probability of having a microscale vorticity $\sigma \leq \omega(\mathbf{x}) < \sigma + d\sigma$ at point $\mathbf{x}$. The macroscopic averaged vorticity is then

$$\overline{\omega}(\mathbf{x}) = \int \sigma n(\mathbf{x}, \sigma) d\sigma. \tag{14}$$

The macrostate is the field of probability over the whole domain. Any microscale distribution of vorticity, which looks on the macroscale like that probability, is said to be a compatible microstate. The fluid is expected to relax to the macrostate which can be achieved in the largest number of ways, and thus is maximally probable. The theory relies on the strong assumption, that the microscale mixing of vorticity is ergodic. This means that the available vorticity is completely free to mix in any possible (area preserving) way, so

that only the probability determines which outcome is likely to be observed. A statistical mechanics canonical approach is undertaken. A free energy

$$F(\{n\}) = -S(\{n\}) - \beta E(\{n\}) + \sum_{l=0}^{\infty} \mu_l Q_l . \tag{15}$$

is maximized. Here $S(\{n\})$ is the entropy function

$$S(\{n\}) = -\int n(\mathbf{x},\sigma) \ln n(\mathbf{x},\sigma) d^2\mathbf{x} d\sigma \, , \tag{16}$$

the energy, expressed in terms of $n(\mathbf{x},\sigma)$, is

$$E(\{n\}) = \frac{1}{2} \int \sigma\sigma' n(\mathbf{x},\sigma) n(\mathbf{x}',\sigma') G(\mathbf{x}',\mathbf{x}) d^2\mathbf{x} d^2\mathbf{x}' d\sigma d\sigma' \, , \tag{17}$$

and the constants of motion are multiplied by appropriate Lagrange factors and added to $F(\{n\})$.

The constraints to be imposed are E = constant, $\int n d\sigma = 1$ and $\int n d^2\mathbf{x} = g(\sigma)$. The function $g(\sigma)$ is the global vorticity distribution, which should be invariant for microscale inviscid flows. To implement the last constraint, the conservation of all moments of the vorticity is instead required. It is assumed to be sufficient that the infinite set of moments of the vorticity are conserved without requiring the topological correctness of the flux. In other words, it is assumed that the area preserving vorticity mappings are dense (at least in the coarse average) in the topologically feasible fluxes. Using $g(\sigma)$ instead of Q_l,

$$\sum_{l=0}^{\infty} \mu_l \int \overline{\omega^l} d^2\mathbf{x} = \sum_{l=0}^{\infty} \mu_l \int \sigma^l g(\sigma) d\sigma = \int \mu(\sigma) n(\mathbf{x},\sigma) d\sigma d^2\mathbf{x} \, . \tag{18}$$

Functional derivation with respect to $n(\mathbf{x},\sigma)$ and algebraic manipulation lead to the system

$$\begin{aligned} n_0(\mathbf{x},\sigma) &= \frac{\mathrm{e}^{-\beta(\sigma\psi_0(\mathbf{x})-\mu(\sigma))}}{\int \mathrm{e}^{-\beta(\sigma\psi_0(\mathbf{x})-\mu(\sigma))}\, d\sigma}, \\ \omega_0(\mathbf{x}) &= \int \sigma n_0(\mathbf{x},\sigma) d\sigma = -\nabla^2 \psi_0(\mathbf{x}) \, , \end{aligned} \tag{19}$$

which has to be solved in order to find the maximally probable macrostate $n_0(\mathbf{x},\sigma)$. In this procedure, the dependencies of β on E and of $\mu(\sigma)$ on $g(\sigma)$ are left as implicit; their actual form is supposed to be found only after solving consistently the system. It is also assumed, but not proven, that Lagrange

multipliers can be determined for any physically accessible values of the conserved quantities. The system (19) does not in fact say very much. For $\beta < 0$, it states that the probability of having at $\mathbf{x}$ a vorticity of the same sign of $\psi_0(\mathbf{x})$ grows with σ, but is shaped by the weight factor $\exp[\beta\mu(\sigma)]$. For $\beta > 0$, the same applies to a vorticity opposite in sign to $\psi_0(\mathbf{x})$. The fact that $\beta < 0$ in physical situations is inferred in comparison with the case of a 2D Coulomb gas (Miller et al. 1992).

Robert and Sommeria (1992) also derive an evolution equation for the approach to the maximum entropy state in this framework. Going further on, Jüttner et al. (1996) propose a way to include random forcing in the maximum entropy theory.

Particular forms of $\psi_0(\omega_0)$ can be found only with additional hypotheses. If, for instance, the vorticity takes only N different values $(\omega_1, \ldots, \omega_N)$, then $n(\mathbf{x}, \sigma)$ will be everywhere a sum of delta functions. The system (19) becomes

$$n_0(\mathbf{x}, \sigma) = \frac{\sum_{i=1}^{N} \mathrm{e}^{-\beta[\omega_i \psi_0(\mathbf{x}) - \mu(\omega_i)]}\, \delta(\sigma - \omega_i)}{\sum_{i=1}^{N} \mathrm{e}^{-\beta[\omega_i \psi_0(\mathbf{x}) - \mu(\omega_i)]}} = \sum_{i=1}^{N} n_i(\mathbf{x}) \delta(\sigma - \omega_i)\,, \tag{20}$$

$$\omega_0(\mathbf{x}) = \frac{\sum_{i=1}^{N} \omega_i\, \mathrm{e}^{-\beta\omega_i \psi_0(\mathbf{x})}\, \mathrm{e}^{\beta\mu(\omega_i)}}{\sum_{i=1}^{N} \mathrm{e}^{-\beta[\omega_i \psi_0(\mathbf{x}) - \mu(\omega_i)]}} = \sum_{i=1}^{N} n_i(\mathbf{x}) \omega_i\,. \tag{21}$$

The latter equation expresses a single valued relation $\omega_0(\psi_0)$. Inverted as $\psi_0(\omega_0)$, it may eventually be multiple-branched. Specifically, for two opposite levels $\omega_1 = -\omega_2$, (21) becomes

$$\omega_0(\mathbf{x}) = \tanh\left(-\beta\omega_1 \psi_0(\mathbf{x}) + \beta \frac{\mu(\omega_1) - \mu(-\omega_1)}{2} \right)\,. \tag{22}$$

The slope of the curve in the (ω, ψ) plane is determined by β, which depends on the energy; the position of the origin is fixed by $\mu(\omega_1) - \mu(-\omega_1)$, which in turn can be expressed as a function of Q (all the higher moments of the vorticity are related to Q by the limitation to two levels).

For a two-level vorticity distribution it is even possible to reconstruct $n(\mathbf{x}, \sigma)$ knowing $\overline{\omega}(\mathbf{x})$. This happens since $n_1(\mathbf{x})$ and $n_2(\mathbf{x})$ are found from

$$n_1(\mathbf{x}) + n_2(\mathbf{x}) = 1\,, \qquad \omega_1 n_1(\mathbf{x}) + \omega_2 n_2(\mathbf{x}) = \overline{\omega}(\mathbf{x})\,. \tag{23}$$

The entropy S can thus be evaluated directly in terms of $\overline{\omega}(\mathbf{x})$:

$$S = \int \frac{[\omega_2 - \overline{\omega}(\mathbf{x})] \ln[\omega_2 - \overline{\omega}(\mathbf{x})] + [\overline{\omega}(\mathbf{x}) - \omega_1] \ln[\overline{\omega}(\mathbf{x}) - \omega_1]}{\omega_2 - \omega_1} d^2\mathbf{x}$$
$$+ \int \ln(\omega_2 - \omega_1) d^2\mathbf{x} \,. \qquad (24)$$

A variety of $\psi_0(\omega_0)$ other than (22) can be derived as well. Assuming *a priori* that $\beta\mu(\sigma) = -|\sigma|/q$, that is, if the weight factor is Poissonian (Pasmanter 1994),

$$\omega_0(\mathbf{x}) = -2 \frac{\beta q^2 \psi_0}{1 - \beta^2 q^2 \psi_0^2} . \qquad (25)$$

Assuming instead that $\beta\mu(\sigma) = -(\sigma/q)^2$, that is, if the weight factor is Gaussian (Miller et al. 1992, section VI D),

$$\omega_0(\mathbf{x}) = -\beta q^2 \psi_0 \,. \qquad (26)$$

A primary importance has been attributed to the 'dressed vorticity corollary' (DVC) (Miller et al. 1992). This corollary says that if one "guesses" n_0 from the macroscopical equilibrium state, that is if one writes $n_0(\mathbf{x}, \sigma) = \delta(\sigma - \omega_0(\mathbf{x}))$, then a frozen dynamics is obtained, which is the $\beta \to -\infty$ limit of the true one. It is to remark that the "dressed vorticity distribution"

$$g_d(\sigma) = \int \delta(\sigma - \omega_0(\mathbf{x})) d^2\mathbf{x} \qquad (27)$$

is, in general, different from $g(\sigma)$. This $g_d(\sigma)$ will be approximated by the histogram of the vorticity distribution computed from a numerical simulation. While $g(\sigma)$ would be conserved by an inviscid dynamics, only $g_d(\sigma)$ may be constructed from the computed field, and will vary in time. Only at the initial time, by definition, $g(\sigma) = g_d(\sigma)$. A consequence of the DVC is that the equilibrium field ω_0 is the one that has maximal energy among all the fields with the same distribution $g_d(\sigma)$.

2.6 Applications of the maximum entropy theory

Several recent papers compare direct numerical simulations with the predictions of the maximum entropy theory. Even recognizing their value, we think that their Ansätze are not justifiable in our cases, or that their conclusions do not match our results. In detail, Miller et al. (1992) take into account a two-valued (0,ω_0) and a three-valued (0,ω_0,$-\omega_0$) initial condition onto a circular corona with no slip boundaries. They solve the variational problem by a Montecarlo dynamics and compare it with a direct numerical simulation of

the flow. Whitaker and Turkington (1994) consider two equal circular vortex patches on a closed disk, with zero ambient vorticity. They use an iterative solver for the constraint equations, and compare the result with more extensive contour dynamics simulations on the infinite plane (not on the disk). Sommeria et al. (1991) consider a shear layer, with periodical boundary conditions in the x direction and slip walls in y. Their simulation starts from a two-level initial condition ($\omega = 0, a$), and the nonlinear eigenvalue problem is solved accordingly. In a particular limit, they are able to compute analytical solutions, which exhibit a bifurcation in the parameter space. One of those branches corresponds to solutions which break the symmetry in x, but are stationary (except for a system of reference mean velocity). Their *a-posteriori* fit of the $\omega(\psi)$ scatterplot is satisfactory only for one tract of the complete curve. They also mention simulations of multiple shear layers, saying that different local vortices are obtained, preventing the system to achieve a steady state ("the system tends to achieve local equilibria into vortices faster than the global equilibrium"). Additional simulations of an idealized jet are done by Thess et al. (1994); also in that case x-symmetry breaking solutions are found from the maximum entropy theory, and a better quantitative match is obtained. Symmetry breaking for periodic square, periodic channel and box boundary conditions is further discussed along those lines in a subsequent paper (Jüttner et al. 1995).

In the maximum entropy setting, continuous symmetries generate additional terms in the exponentials of the equations (19). If these terms involve an explicit dependence on the coordinates, such as in the case of the conserved angular momentum on the infinite plane, the soultion of (19) could be non stationary (Robert and Sommeria 1991). In the periodic case, however, no continuous symmetry besides the translation exists, and this possibility is prevented.

Chavanis and Sommeria (1996) start assuming that $\omega(\psi)$ is linear, as it is in the minimum enstrophy context, and give analytical maximum entropy solutions for rectangular and circular closed domains. This is said to be justifiable in a particular 'strong mixing' limit. The solutions are always stationary, but admit monopole/multipole bifurcations, which are thoroughly listed.

A true attempt to validate the maximum entropy rather than the minimum enstrophy theory in an experiment is done by Huang and Driscoll (1994). They consider a rather simple metaequilibrium profile of a magnetized electron column, which obeys to the 2D Euler equation. They conclude that the mixing cannot be assumed to be ergodic and that the closest fit to the data is provided by the numerical minimum enstrophy solution.

3 Numerical experiments

We performed a number of numerical experiments. To this extent we integrated in time several vorticity fields, using a rather standard protocol for the Navier–Stokes equation. We used a two-dimensional 2/3 dealiased pseudospectral code on the periodical square $(2\pi)^2$. This choice, which fixes the Green function of the problem, was done for easiness of implementation. Nothing in the previously exposed theories prevents us to use this particular choice of boundary conditions. The Green function becomes $G(\mathbf{k}) = 1/k^2$ in Fourier space, as known. The (undealiased) resolution of the vorticity fields considered ranges between 16^2 and 512^2. A small viscosity is introduced mainly for numerical purposes, in order to prevent finite size effects, such as high wavenumber pile-up. Viscosity is seen to be too small if the energy, as seen from the $E(k)$ spectrum, accumulates at high k. According to a generally accepted practice, we used hyperviscous dissipative terms of the form $\nu_2\nabla^4\omega$ or $\nu_3\nabla^6\omega$ in place of the ordinary viscous term. Even if drawbacks in the use of hyperviscosity instead of normal viscosity have been recently discussed (Jimenez 1994), we think that the choice is in practice uninfluent to our simulation. We confronted runs in which the same initial conditions were integrated with different forms of the dissipative term, observing almost no influence on the resulting final configuration. The small scale character of the hyperviscosity suppresses also some truncation effects, such as Gibbs wiggles in the proximity of steep gradients of the vorticity. Some dissipation is needed to 'underresolve smoothly' the smallest features which form during the evolution. The amount of dissipation has to be grossly matched with the rate of creation of smallest-scale structures, but does not have to be fine tuned. Time marching is accomplished by a fourth order Runge-Kutta integrator, with explicit treatment of the dissipative term. A Courant–Friedrichs–Levy condition is employed to vary the time-step. The adoption of $\Delta t = 0.3\Delta x/|\mathbf{v}_{\max}|$ appears to be accurate enough.

Most of our runs started from initial conditions consisting of variously arranged constant patches of vorticity. In most cases we simply distributed equal areas of vorticity equal to ± 1 on the square. This already provided us with quite a variety of outcomes, and allows direct connection with the various formulas of section 2. A pattern of this sort is the family of the 'fuzzy checkers'. By these we mean checkers with randomly perturbed edges. The perturbation is done in a way that insures a constant zero mean vorticity. A regular checker of vorticity is an unstable stationary field (the velocity is orthogonal to the gradient of vorticity, which is significantly different from zero only on the boundaries), while 'fuzzy' checker is immediately destabilized. Another possibility to generate (usually) unstable initial conditions, is to consider random arrangements of rectangular $\omega = \pm 1$ tiles upon the domain. A little smoothing on the initial conditions is actually necessary to prevent the edge ringing mentioned above, but, again, it does not seem critical for the results. As re-

Fig.	res.	ν_3	T_f	E_0	E_f	Q_0	Q_f	$\diamond$	$\equiv$
1	400^2	$4 \cdot 10^{-13}$	449	.01078	.01072	.457	.015	+14%	−6.1%
2	256^2	$2 \cdot 10^{-11}$	745	.1052	.1051	.455	.107	+.26%	−10%
	64^2	$4 \cdot 10^{-8}$	660	.119	.117	.36	.12	+.17%	−11%
	256^2	$2 \cdot 10^{-11}$	741	.0138	.0135	.48	.018	+2.7%	−19%
	64^2	$4 \cdot 10^{-8}$	1047	.034	.032	.319	.037	−.04%	−20%
	64^2	$1.9 \cdot 10^{-8}$	1291	.0107	.0082	.431	.012	+2%	−30%
	256^2	$2 \cdot 10^{-11}$	343	.16458	.16454	.29	.20	+.6%	−16%
4	128^2	$2 \cdot 10^{-9}$	704	.0178	.0169	.386	.022	+1.8%	−14%
	256^2	$2 \cdot 10^{-11}$	219	.16417	.16406	.427	.169	−.5%	−2%
	256^2	$5 \cdot 10^{-12}$	267	.1329	.1328	.466	.144	−1.2%	−4.8%
	64^2	$4 \cdot 10^{-8}$	182	.3963	.3961	.445	.433	−12%	+1.4%
5	256^2	$2 \cdot 10^{-8}$	165	.1639	.1629	.418	.184	−8%	+5.9%
	256^2	$2 \cdot 10^{-9}$	175	.1639	.1634	.418	.184	−9.4%	+4.3%
	256^2	$2 \cdot 10^{-10}$	228	.1639	.1637	.418	.186	−8.6%	+5.2%
	256^2	$2 \cdot 10^{-11}$	293	.1639	.1638	.418	.187	−11%	+3.5%
7	512^2	$4 \cdot 10^{-12}$	340	.1991	.1991	.486	.213	−4.5%	+3.7%
	256^2	$2 \cdot 10^{-11}$	208	.1991	.1990	.487	.213	−4.8%	+3.5%
	64^2	$4 \cdot 10^{-8}$	367	.199	.198	.486	.202	−3.5%	+1.1%
	32^2	$2 \cdot 10^{-7}$	419	.199	.196	.467	.199	−1.5%	−.2%
	16^2	$1 \cdot 10^{-6}$	907	.198	.190	.408	.191	−1.3%	−3.9%

Table 1
Resolution, hyperviscous coefficient ν_3, total integration time T_f, initial and final values of energy and enstrophy E_0, E_f, Q_0, Q_f for the runs presented. The last two columns give the increase in energy of the "prototype rearrangements" described in Appendix A.3, relative to E_f. $\diamond$ stands for the 'square dipole' and $\equiv$ for the 'smooth stripe'.

marked above, those initial conditions, even if a little odd in appearance, are perfectly legitimate as test cases for the relaxation models. The resolutions of the various runs presented, together with the hyperviscous coefficient ν_3, the total integration time T_f, the initial and final values of energy and enstrophy E_0, E_f, Q_0 and Q_f, are reported in Table 1.

A typical case is shown in figure 1. The patches initially intertangle in a complicated way. The process generates a lot of filamentary features, which, both because of the finite resolution and the small-scale dissipation, are blurred out

and disappear from the landscape. At later times, the final shape becomes apparent. The process of stretching and uniformization of the details continues, and a 'simpler' state stabilizes. This configuration undergoes no further evolution, apart of a slow erosion of its contours, due to the viscosity. Since the viscosity can in principle be made very small, this state is long lived and can be named 'final'. At this stage, the viscous term of equation (1) is some orders of magnitude smaller than the vorticity (or the strain) field, everywhere on the domain. The evolution of the velocity field is therefore still dominated by the nonlinear terms, which balance to make the convective term null and the vorticity configuration stationary. The timescale for viscous damping is much longer than the characteristic turnover time.

In the case of figure 1 the final state is a dipole. Something similar is also observed in other cases. We document a few of them in figure 2. The broadest vorticity contours in the final state appear squared because of the biperiodic boundary conditions. This reflects the shape of the Green function (Glasser 1974,Seyler, Jr. 1976). The maximum and the minimum of vorticity are unique, and displaced each other by half of the box size in both coordinates. The final state is stationary, and this agrees with the traditional point of view, but not entirely. The profiles of the positive and negative cores may differ. The scatterplot of (ω, ψ) tends to a line, but does not match any of the expected functional relations, such as (8), (13) or (22). In all cases, the scatterplot of (ω, ψ) is seen to lie in the first and third quadrants. This is consistent with any of the proposed $\omega(\psi)$ relations, in which β is negative. In addition, the different cases mix differently the available vorticity. If we plot the global distribution of the vorticity (third panels of the graphs in figure 2), we see that the final histograms vary from case to case. The profile of the final vortex cores is therefore peculiar to each decay. In all of them, however the final vorticity has a single maximum and sigle minimum, and the dipole may be thought as an arrangement of the avaliable vorticity around two centers.

The final dipole is not the only possibility, though. In figure 3, we see a different outcome. In this case a pattern of two stripes is formed. The boundaries of these two stripes translate in opposite directions varying slightly their curvature according to their relative position. The state is therefore recurring. Interestingly, the two zones cannot be explained simply by the absence of mixing of the original patches: detailed analysis shows that each stripe contains a fraction of the fluid of originally opposite vorticity. The mixing inside a single stripe is complete, so that the vorticity is almost constant and lower than that of the likely-colored patch in the initial condition. The effectiveness of mixing within the stripe, but its absence among the two stripes, is clearly evidenced when including numerical passive markers into the flow (figures omitted).

In figure 4 we see other examples of fields which decay in nonstationary final

states. There is a variety of forms: non-symmetric fat cores, which oscillate in time; wavy stripes with central blobs that translate in one direction. Yet the last case shown in figure 2 is not really a simple dipole; actually three smaller and long lived vortices happen to form inside the main core. They keep orbiting for long time and weaken very slowly because of the viscosity, but are never able to merge. It is sometimes common, as in the case of figure 3, to get a complete uniformization of the vorticity in different well-separated zones of the flow. A number of patches of almost constant vorticity are thus formed, and their arrangement prevents further mixing. In all these cases the scatterplot of (ω, ψ) never thins out, and, *a fortiori* cannot be fitted by any of the relations given in section 2. Nevertheless, these configurations are long lived. Several features are still common to both types of decays. The final states are "dipoles" in a broad sense, in that they show two simply connected regions of vorticity above and below the mean. The local maxima of $|\omega|$ are also absolute maxima. The small-scale structures are generated (and subsequently blurred) quite early in the simulation. If we compute characteristic eddy turnover times based on the sizes of the patches in the initial conditions, we see that the palinstrophy (associated to the amount of details) reaches its maximum within a few turnarounds. The entire simulation is carried on typically for some tens of turnover times. The loss of energy during the decay is always negligible. The enstrophy is seen to decay significantly during the decay, until the final state is formed; it then remains nearly constant. Most of the theories proposed in section 2 take into account a lowering of Q, but fix it in an unique way. We see that not even the initial E *and* Q are sufficient to determine the final state: as a counterexample, the case of figure 3 and the one shown in the second row of figure 4 possess initially nearly the same values of E and Q, and the same distribution of vorticity, but reach different final states. Other counterexamples may be easily generated taking an intermediate configuration and reversing the time integration. The blurring of details is an irreversible process, and the backward integration does not lead back to the initial condition. The forward and backward final states have generally different properties.

For the initial conditions we have used, the final states appear to be quite independent from the resolution and the (small) values and forms of the viscosity. Generally, a higher spatial resolution just requires a longer time to finish to smooth the details and to reach the final state. To ground this affirmation we give some examples.

First, we repeated the simulation shown in figure 3 with four different values of the hyperviscosity. The appearance of the field after long times is shown in figure 5. The same wavy pattern, with uniform stripes of vorticity and undulated boundaries is seen. Phase differences between the final panels are not meaningful, since the field is plotted at different times. To demonstrate the convergence and to provide one example of the evolution in time of the

characteristic quantities, we plot, in figure 6, $E(t)$, $Q(t)$ and $P(t)$ for the different values of ν_3. The functions E and Q always stabilize on the final level. The loss in E is indeed negligible, less than 1% in the most dissipated case. The enstrophy instead decreases of roughly a factor two, but stabilizes to a final value independent of the hyperviscosity. The palinstrophy is seen to increase in time in the initial stages. This behavior, which is also predicted by closure theories' estimates, is related to the increase of small-scale details. The maximum value is achieved when such details begin to be more rapidly numerically underresolved than generated. This can be seen by looking at the field in physical space (compare for instance the intermediate panels of figure 3). The curves are also seen to scale inversely to the hyperviscosity. This behavior, apart from different values and different relaxation times, is also found in all the other decays, both the stationary and nonstationary ones. It is also observed when normal viscosity or second viscosity are employed.

Secondly, in order to show that the final state found is not an artifact of the finite resolution, we display the integration of another arbitrary initial condition at different spatial resolutions (figure 7). The hyperviscosity coefficients are adjusted case by case according to the highest wavenumber retained. We see that unless the resolution is extremely poor (16^2), a final state with given characteristics is obtained. In this case, the final state is nonstationary, the two elongated cores translate in one direction, while both background stripes convect in the opposite one.

We note, passing by, that the Fourier spectrum of these final states is always decreasing in k. We are not however concerned with possible laws to fit their slopes, and we do not report them here. In general, the final state is neither a pure $|\mathbf{k}| = 1$ state (as predicted by the minimum enstrophy principle), nor a spectrum introduced in section 2.1. Most of the energy of the final field is contained in the first modes, but this fact itself does not explain the formation of a huge variety of profiles.

4 Conclusions and perspectives

We have considered several examples of relaxations of two-dimensional vorticity fields, and compared them with the theories which have been so far proposed in literature. We have brought some numerical evidence that contrasts with the existing final states theories. The different theories do not agree in their predictions, as can be seen in the different $\omega_0(\psi_0)$ which they propose. We have shown the existence of legitimate final states which are not included in the accepted point of view. Nonstationary ones are among them. The remark that stable, nonstationary configurations may indeed form from unstable ones is indeed not new. Stable tri- and quadrupole satellite systems, which result from

the decay of unstable vortex blobs, have already been investigated analytically and numerically (Carton et al. 1989,Carton 1992,Morel and Carton 1994,Carton and Legras 1994). Beautiful experimental evidence is also provided (van Heijst et al. 1991,Flor and van Heijst 1996). In all these cases (ω, ψ) is derived from the experiment, and is seen to evolve from a non-monotonous to a branched monotonous curve. On the other hand, also the non-uniqueness of stable dipolar solutions on the periodic box has been stressed (Rasmussen et al. 1996). Numerical experiments shown there display (ω, ψ) scatterplots which are tentatively fitted by polynomial relations. For this reason we have not tried to fit one or the other of the free parameters of the reported models versus our numerical results: the possibility of a good fit appears more accidental than general. In our opinion many of the underlying assumptions of these theories are not always granted, and call therefore for a more careful treatment. For instance, we think that the final state cannot be predicted by a 'final state theory' which ignores completely the dynamical path underwent by the relaxing system. This is demonstrated by the fact that configurations with the same two initial vorticity levels decay to different final states. The quest for appropriate statistical descriptors, which are preserved during the evolution, is still open. For instance, it is not known how the initial macrovorticity distribution relaxes to the final one. It would be appealing to parametrize this change by a few dynamical quantities, such as the decrease of Q. We have supposed that some of the shapes of the final states can be understood as rearrangements of the vorticity which is available at late times. This seems to apply at least to the final stationary states, for which the requirement of energy maximization appears as sufficient. The instability of the intermediate vorticity configurations is always such that the system is led to mix its vorticity and drawn toward the final state. In the other cases we may conjecture that, for some stability reason, the available vorticity is unable to collapse onto a stationary configuration. In many of the previous works, initial Gaussian fields with random spectra were considered. These fields show generally a smaller population of absolute higher vorticity (i.e., $g_d(\sigma)$ with long tails). It is often observed (but not always) that a lot of concentrated vorticity patches are formed in the early times of evolution and they survive relatively long times (see e.g. Brachet et al. 1988,Kida 1985). Such intermediate states, composed of several isolated cores, which have so often been described in literature, could be understood as metastable preliminary local rearrangements of the vorticity.

In other cases, however, the attainment of stationary states seems prevented. Possibly, the system is trapped in a metastable state, which is not stationary (in the cases encountered in this work, periodical in time). The stationary state could be missed just because the dynamics of the vorticity is by itself insufficient to trigger a catastrophic mixing process, which alone could alter further the vorticity distribution, and lead the system to its very final state. It would then be very interesting to find the requirements for this metastability. An implication of such metastabilities would be that some kind of external

forcing might trigger the transition to the stabler states and some perhaps not.

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Appendix: Energy maximization and stability

All the theories proposed in section 2 identify the final state as the one which solves a certain extremum problem. In the case of the minimum enstrophy principle, one may think to rescale the vorticity, and to rephrase the problem as to the search for the maximal energy, given a fixed enstrophy. In the setting of the minimum entropy, the DVC instead tells us that the final state is actually also a state of maximal energy (with fixed g_d). It is therefore worth to add some remarks on energy maximization.

We are not able to determine in how many equally energetic ways the same amount vorticity can be distributed over a domain. We might ask if equally energetic states which are not dynamically related, i.e., which do not evolve one into the other, because they are topologically different, could converge to the same final state. We cannot answer to those questions, but we can add a few considerations which at least rule out some possibilities. These provide necessary requirements, but they do not determine the configuration of vorticity nor the flow.

A.1 Infinitesimal deformations

Let us first consider a generic area preserving infinitesimal deformation on a two-dimensional domain. The deformation may be written as $\delta\mathbf{x} = \nabla^{\perp}\delta\phi(\mathbf{x})$,

where ϕ is an arbitrary field. It is easily seen that such transformations preserve all the moments of the vorticity:

$$\begin{aligned}\delta \int \omega^l d^2\mathbf{x} = & l \int J(\omega, \delta\phi)\omega^{l-1} d^2\mathbf{x} = \\ & = -l \int \delta\phi \left[(\omega_x \omega^{l-1})_y - (\omega_y \omega^{l-1})_x\right] d^2\mathbf{x} = 0,\end{aligned} \tag{28}$$

Integration by parts is carried out since the boundary terms vanish for any boundary condition. The variation of the energy with respect to such deformations is

$$\begin{aligned}\delta E = & \delta \tfrac{1}{2} \int \omega(\mathbf{x})\omega(\mathbf{x}')G(\mathbf{x}', \mathbf{x}) d^2\mathbf{x} d^2\mathbf{x}' = \int \delta\omega(\mathbf{x})\omega(\mathbf{x}')G(\mathbf{x}', \mathbf{x}) d^2\mathbf{x} d^2\mathbf{x}' = \\ = & - \int J\left(\delta\phi(\mathbf{x}), \omega(\mathbf{x})\right) \omega(\mathbf{x}')G(\mathbf{x}', \mathbf{x}) d^2\mathbf{x} d^2\mathbf{x}' = \\ = & \int \delta\phi(\mathbf{x}) J\left(\omega(\mathbf{x}), G(\mathbf{x}', \mathbf{x})\right) \omega(\mathbf{x}') d^2\mathbf{x} d^2\mathbf{x}' \, .\end{aligned} \tag{29}$$

Therefore the extremal energy states satisfy

$$\frac{\delta E}{\delta \phi} = \int J\left(\omega(\mathbf{x}), G(\mathbf{x}', \mathbf{x})\right) \omega(\mathbf{x}') d^2\mathbf{x}' = J\left(\omega(\mathbf{x}), \psi(\mathbf{x})\right) = 0. \tag{30}$$

Thus if the dissipation is absent, the stationary states extremize the energy with respect to infinitesimal incompressible deformations. These states are "local" extrema.

A.2 Pointwise mixing exchanges

We can write a transformation which transfers some of the vorticity present in the neighborhood of $\mathbf{x}_1$ to a similar neighborhood of $\mathbf{x}_2$ and vice versa. To this extent we employ a shape function $\delta_\epsilon(\mathbf{x})$, which is equal to 1 in a neighborhood of radius ϵ of $\mathbf{x} = 0$ (modulo the boundary conditions) and zero everywhere else. The vorticity field is transformed according to

$$\begin{aligned}\omega(\mathbf{x}) \to \omega(\mathbf{x}) & + r\left[\omega(\mathbf{x} - \mathbf{x}_1 + \mathbf{x}_2) - \omega(\mathbf{x})\right] \delta_\epsilon(\mathbf{x} - \mathbf{x}_1) \\ & + r\left[\omega(\mathbf{x} + \mathbf{x}_1 - \mathbf{x}_2) - \omega(\mathbf{x})\right] \delta_\epsilon(\mathbf{x} - \mathbf{x}_2) \, .\end{aligned} \tag{31}$$

The parameter r may vary from 0 to 1; in the latter case, an exchange between the vorticity in $\mathbf{x}_1$ with that in $\mathbf{x}_2$ is realized. The energy of the transformed field is computed by substituting the above expression in equation (3). When consider the limit $\epsilon \to 0$, the integrals are approximated by the mean value

of the integrand times the area of the integration region, provided that ω is continuous. The terms multiplied by r come of order $O(\epsilon^2)$, while the ones with r^2 remain $O(\epsilon^4)$. The singularity in the Green function poses no problems, since for small arguments $G(\mathbf{x}, \mathbf{y}) \sim -\ln|\mathbf{x}-\mathbf{y}|$, which is integrable. Therefore

$$E \to E + r\pi\epsilon^2[\psi(\mathbf{x}_2) - \psi(\mathbf{x}_1)][\omega(\mathbf{x}_1) - \omega(\mathbf{x}_2)] + r^2 O(\epsilon^4)\,. \tag{32}$$

This gives us the infinitesimal change in energy due to a transformation which modifies infinitesimally, but not continuously, the field. The fact that this transformation is not dynamical is not of our concern, as far as we look for properties of the configurations and not for their evolution.

If we refer to the maximal energy state, its energy shall decrease whichever the couple of points $\mathbf{x_1}$ and $\mathbf{x}_2$ and the value of r. Therefore, $\omega(\mathbf{x}_2) > \omega(\mathbf{x}_1)$ implies $\psi(\mathbf{x}_2) > \psi(\mathbf{x}_1)$ and vice versa. The most important consequence of it is that $\psi(\omega)$ must be single valued and monotonous. This is seen by considering the plot of $\omega(\mathbf{x})$ versus $\psi(\mathbf{x})$. The monotonicity is implied by the fact that data points on the (ω, ψ) plane can be labelled by $\mathbf{x}$. If the relation $\omega(\psi)$ was non monotonous or more than single valued, then it would be possible to find points at which $\omega(\mathbf{x}) \leq \omega(\mathbf{x}_1)$ and $\psi(\mathbf{x}) \geq \psi(\mathbf{x}_1)$, contrary to the hypothesis. As a further consequence, the absolute maximal energy state is also stationary, because $\omega(\mathbf{x})$ and $\psi(\mathbf{x})$ are functionally related.

A particular case is achieved when the points are infinitesimally close and displaced in the direction $\mathbf{d}$: in this case (32) is read as $(\mathbf{d} \cdot \nabla\psi)(\mathbf{d} \cdot \nabla\omega) > 0$ at any point and along any direction.

Related to this, but derived with a different machinery, is the Rayleigh–Arnold criterion (Holm et al. 1985,McIntyre and Shepherd 1987,Carnevale and Vallis 1990). It states that among all the isovortical fields the maximal energy arrangement is unique and stationary. Furthermore, if $d\psi_0(\omega_0)/d\omega_0$, or equivalently $-\nabla\psi_0 \cdot \nabla\omega_0/|\nabla\omega_0|^2$ are strictly positive and limited, then the field is also nonlinearly stable. We also note that there may exist also other stable states, in particular those obtained by means of some symmetry transformation, which is specified by invariants inexpressible as functionals of the vorticity. Analytical stability criteria other than Arnold's one, that refer to local features in ambient flows (e.g. Nycander 1995), do not appear to be of direct use for assessing global stability.

A.3 Energy maximization algorithms

We have seen in A.1 and A.2 properties of 'energy-maximizing arrangements of the available vorticity'. By 'arrangement', we mean another field, which

has exactly the same fractional area for each level of vorticity (i.e. the same vorticity histogram), but a different configuration. The preceding facts tell us that these rearrangements are unique, stationary and stable. We are not, however, guaranteed that *any* initial field relaxes into such an energy maximizing configuration. Not only it is *a priori* undetermined how the histogram of macrovorticity evolves in time, but also not all fields are seen to approach a stationary final state. We want therefore to characterize the maximal energy state for a given vorticity population, and its relation with the observed dynamical final state.

One possibility would be provided by the Carnevale, Vallis and Young pseudodynamics (Vallis et al. 1989,Carnevale and Vallis 1990). They proposed a numerical procedure to deform a vorticity field by increasing its energy and conserving its enstrophy. The procedure does not explain the geometrical properties of the extremal states, which can only be said to be local, and not absolute maxima of the energy. Lacking analytical procedures, we rather used a simpler alternative combinatorial approach. Starting from a given configuration of vorticity, two square tiles are randomly exchanged and the new energy is computed. The new configuration is kept if the energy increases, discarded otherwise, and the process repeated. Miller et al. (1992) also use a more sophisticate numerical procedure of this kind. Such approach is easy to implement, but very slow in convergence when the number of tiles is significant. With the aid of this tool, we observed that the highest energy configuration achieved by permutation of equal area tiles of vorticity $\pm\omega_1$ on the periodic square, is a subdivision in two stripes parallel to the sides. When intermediate levels of vorticity are present, as in our runs, where patches' edges are significantly smoothed, the most energetical of these permutations of tiles has sometime still a striped shape, sometimes a central cored appearance.

To cope with high resolution rearrangements, we chose to limit ourselves to compare energies of only a few selected prototypical configurations. To produce them we sorted the discrete tiles of the field according to value of their vorticity, and deployed them upon the domain following the same order of a sample configuration. In this way we obtain a rearrangement with the same set of contour lines of a given prototype. We can then check which, among few prototype rearrangements, increases the current energy. One of the test profiles we chose is the "square dipole", which is modeled on $G(\mathbf{x})-G(\mathbf{x}+(\pi,\pi))$; its contour lines look similar to those of various dipolar final states shown in Fig. 2. Another one is the parallel stripe, which is the maximal arrangement for a two-level population. If we compute the energies of the rearrangements of the final states obtained in section 3, we basically see that for the stationary final states the highest energy is achieved for the 'square dipole' rearrangement, with minimal increase of energy. For the nonstationary final states, such as the wavy ones, often the stripe rearrangement results more energetical. The actual values of the energies are reported in Table 1. The choice of the max-

imal profile is basically a function of the available vorticity. Still, the energy of the nonstationary states is often significantly increased by the procedure. This indicates that the final state, in those cases, is far from the highest energy configuration for the given g_d. Conversely, the final state may be dynamically inaccessible from the rearrangement: such is the case for the wavy configurations, if we maintain that the energy is almost conserved. In addition, the wavy cannot be seen as a perturbed stripe, since the maximal energy state cannot be unstable to transversal perturbations, thanks to the Arnold stability criterion.

We are not able to provide a full analytical treatment of the family of the recurring wavy solutions here, but we would like to add a small piece of numerical evidence. We plot the energy corresponding to all two level striped fields with sinusoidal undulations, with varying amplitudes and phase shift (figure 8). We see that the energy has to decrease significantly with increasing amplitude in order to allow for undulations of the boundaries. For a range of energies below the highest, energy isolines span the entire interval $0 \le \Delta\phi \le 2\pi$, with the amplitude depending on the phase shift. A recurring evolution of the field, with pulsating amplitudes, is therefore allowed. Without attempting to make a precise fit to our final states, we claim that the picture is appropriate. It is clear from the figure that for amplitudes of the undulation higher than $A \sim \frac{1}{4}\pi$, but still lower than the geometrical limit $A = \frac{1}{2}\pi$, the range $0 \le \Delta\phi \le 2\pi$ is no more entirely accessible at constant E (the separatrix is drawn in bold). In fact, numerical tests with initial conditions of this kind exhibit stability below an amplitude threshold, and a catastrophic mixing above.

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Fig. 1. A typical case: decay of a two-level patched vorticity field into a stationary dipole. Plots of the vorticity field at various times. A few vorticity isolines are drawn for reference.

Fig. 2. Arbitrary 2-level initial conditions which decay in final dipoles. Each row illustrates one case, with, in each panel: the initial and the late time vorticity field; the initial (dashed line) and final (full line) global vorticity distributions; the final scatterplot of (ψ,ω). All cases start from initial patches with $\omega = \pm 1$, except for the last one, where $\omega = -1.5, +0.5$. The mean vorticity is always zero.

Fig. 3. A less typical case: decay of a two-level patched vorticity field into a wavy pattern.

Fig. 4. Arbitrary 2-level initial conditions which decay in nonstationary configurations. The format of presentation is the same as in Fig. 2.

Fig. 5. Decay to the final wavy state, obtained with different hyperviscosities. The format of presentation is the same as in Fig. 2. The value of ν_3 is respectively $2 \cdot 10^{-8}$, $2 \cdot 10^{-9}$, $2 \cdot 10^{-10}$, and $2 \cdot 10^{-11}$.

Fig. 6. Energy, enstrophy and palinstrophy decay for different values of the hyperviscosity.

Fig. 7. Effect of lowering the resolution of the simulation. The same initial condition is integrated at 512^2, 256^2, 64^2, 32^2, 16^2, and hyperviscosity rescaled appropriately. The format of presentation is the same as in Fig. 2.

Fig. 8. Energy of a family of wavy patterns of vorticity equal to ± 1, as a function of A and $\Delta\phi$. The region of the $(A, \Delta\phi)$ plane where the two stripes would interfere is drawn cross-hatched.